PUTTING DIFFERENTIAL AND INTEGRAL CALCULUS INTO CONTEXT
Blog #6

Steve Strogatz, in his Opinionator column in the April 18, 2010, issue of *The New York Times* published an article on differential and integral calculus with the title *It Slices, It Dices.*

High school and undergraduate mathematics hardly do justice to calculus. They get lost in the techniques of calculation, without revealing the underlying great ideas. To see the deeper interrelation between the differential (the derivative), the indefinite integral (the anti-derivative) and the definite integral, and to put things into their right context, we must step out of one dimension and make ourselves at home in higher dimensional graded vector spaces. Then the beautiful panorama of the world of integrals opens up, exhibiting the symmetry about which the one-dimensional case reveals nothing.

The grading of vector spaces involved is furnished by the degree of differential forms that are to be integrated, and by the dimension of the domain over which integration is to take place. The integral manifests itself as the scalar product $\langle \alpha, \omega \rangle$ of the domain α and the differential form ω where the dimension of α and the degree of ω must agree.

The scalar product is bilinear, i.e., it is linear in each of the two variables:

$$\langle \alpha+\alpha', \omega \rangle = \langle \alpha, \omega \rangle + \langle \alpha', \omega \rangle, \quad \langle n\alpha, \omega \rangle = n\langle \alpha, \omega \rangle$$
$$\langle \alpha, \omega+\omega' \rangle = \langle \alpha, \omega \rangle + \langle \alpha, \omega' \rangle, \quad \langle \alpha, r\,\omega \rangle = r\,\langle \alpha, \omega \rangle$$

(n is an integer and r is a real number).

We also have a dual pair of operators ∂ and d. The boundary operator ∂ assigns to the domain α its boundary $\partial\alpha$ of dimension one lower; the differential operator d assigns to the differential form ω its derivative $d\omega$ of degree one higher. These operators satisfy the relations $\partial\partial = 0$ and $dd = 0$.

If the differential form ω has an anti-derivative φ, i.e., $\omega = d\varphi$, then ω is called an exact differential form. In this case the Fundamental Theorem of Calculus applies. It states that the operators ∂ and d are conjugates of one another, in formula:

$$\langle \alpha, d\varphi \rangle = \langle \partial\alpha, \varphi \rangle$$

where $\partial\alpha$ is the boundary of α and φ is the anti-derivative of $d\varphi$. The dimension of the domain α is n; the degree of the differential form φ is $n-1$.

Let us now see how the familiar special cases arise from the general concept.

(1) The case for functions of one real variable:

$$\int_a^b f(x)dx = F(b) - F(a)$$

The domain α is a line interval with endpoints a, b; the differential form ω is $d\varphi = f(x)dx$; $\partial\alpha$ is the pair of endpoints of the interval from a to b; $\varphi = F(x)$ is the antiderivative of $d\varphi$. We should recognize that the right-hand side $F(b) - F(a)$ is also an integral: that of $F(x)$ over the pair of points a, b.

(2) Stokes' Formula: $\iint_\alpha \operatorname{curl}\omega = \int_{\partial\alpha} \omega$. The domain α is a piece of a surface in 3-space; $\partial\alpha$ is the closed curve in space that serves as the contour for α; $d\omega = \operatorname{curl}\omega$.

(3) Gauss' Formula: $\iiint_\alpha \operatorname{div}\omega = \iint_{\partial\alpha} \omega$. The domain α is a solid in 3-space; $\partial\alpha$ is the closed surface that serves as the boundary for α; $d\omega = \operatorname{div}\omega$.

Indefinite integral

There is a semantic confusion arising out of the use of the term "integral" in two entirely different senses: the definite and the indefinite integral. Given an exact differential form ω of degree n, its anti-derivative, also known as indefinite integral, is the differential form φ of degree $n-1$ such that $\omega = d\varphi$.

Differential forms of degree 1 are of the form $f\,dx$ where f is a function of a real variable. They are exact and their indefinite integrals $F(x)$ are differential forms of degree 0, i.e., ordinary functions. For differential forms of degree 0 we have: $dF = 0 \Leftrightarrow F(x) = C$, constant.

Domains of dimension 1, over which differential forms of degree 1 are to be integrated, are arcs of (space) curves. Their boundary is the pair of endpoints *a*, *b*. Ordinarily by an integral is meant the integral of a differential form of degree 1 over a domain which is a straight line segment. We can now see the whole spectrum of integrals of higher dimensions, and the wonderful symmetry they represent.

The anti-derivative or the indefinite integral, as its name suggests, is not uniquely determined. In the special case of a differential form of degree 1 we can get all the anti-derivatives from a given one by adding a constant *C*. In the general case the rule is not as simple. Let φ and φ' be differential forms of the same degree. The necessary and sufficient condition that they have the same indefinite integral, in other words, φ and φ' are cohomologuous, is as follows:

$$\varphi \equiv \varphi' \Leftrightarrow d(\varphi - \varphi') = 0$$

The equivalence class of differential forms represented by φ is called the cohomology class of φ. We shall denote the quotient set of all cohomology classes by the symbol *W*; it is a graded vector space. Strictly speaking when we talk about differential forms, we always mean their cohomology classes.

The cohomology class of differential forms of degree 0 represented by the function $f(x)$ is the set of all functions $f(x) + C$, where *C* is constant.

Definite integral

Similarly, when we talk about domains of integration, we really mean homology classes. The meaning of homology is as follows. Let α and α' be two domains of integration of the same dimension. We raise the question under what circumstances will the differential form ω have the same integral over the two different domains? The necessary and sufficient condition for α and α' to be homologuous is this:

$$\alpha \equiv \alpha' \Leftrightarrow \partial(\alpha - \alpha') = 0$$

The equivalence class of domains represented by α is called the homology class of α. We denote the quotient set of all homology classes by the symbol *A*; it is also a graded vector space. Strictly speaking, when we talk about domains of integration, we always mean their homology classes. The concept of a definite integral refers to the dual pairing of the graded vector spaces A and W through the scalar product $\langle \alpha, \omega \rangle$. In more detail, the definite integral of ω (a cohomology class of differential forms) over α (a homology class of domains), where the degree of ω agrees with the dimension of α, is a real number that can be obtained as a scalar product. It only depends on the homology class of the domain α and the cohomology class of the differential form ω:

$$\partial(\alpha - \alpha') = 0 \;\Rightarrow\; \langle \alpha, \omega \rangle = \langle \alpha', \omega \rangle$$
$$d(\omega - \omega') = 0 \;\Rightarrow\; \langle \alpha, \omega \rangle = \langle \alpha, \omega' \rangle .$$